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A PRODUCT FORMULA FOR SPHERICAL REPRESENTATIONS OF A GROUP OF AUTOMORPHISMS OF A HOMOGENEOUS TREE, I

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Abstract. Let $G = \mathrm{Aut}(T)$ be the group of automorphisms of a homogeneous tree T, and let Γ be a lattice subgroup of G. Let π be the tensor product of two spherical irreducible unitary representations of G. We give an explicit decomposition of the restriction of π to Γ. We also describe the spherical component of π explicitly, and this decomposition is interpreted as a multiplication formula for associated orthogonal polynomials.

1. Introduction and notation

Let G be the group of automorphisms of a homogeneous tree T. We fix a vertex o of T, and let $K = \{g \in G : go = o\}$. As G is a type I group [3, p. 112], each continuous unitary representation π can be written in an essentially unique way as a direct integral $\int_{\hat{G}} \sigma \, dm(\sigma)$ [5, Thms. 2.15, 1.21]. Now (see [3]) $\hat{G}$ consists of the equivalence classes of (a) the *spherical* irreducible unitary representations (those having nonzero K-invariant vectors), (b) two *special* representations, and (c) an infinite sequence of *cuspidal* representations. The representations in (b) and (c) make up the discrete series of G. Thus the representation space H_π of π can be decomposed as an orthogonal direct sum

$$H_\pi = H_1 + H_2, \tag{1.1}$$

of π-invariant subspaces H_1 and H_2, where H_1 is the closed linear span in H_π of the set of vectors $\pi(g)\xi$, where $g \in G$ and where ξ is K-invariant. The H_1 component π_1 of π must be a direct integral

$$\pi_1 = \int_{\hat{G}} \sigma \, dm(\sigma), \tag{1.2}$$

where m is supported on the spherical part of $\hat{G}$, while the H_2 component π_2 of π must be a direct sum

$$\pi_2 = \sum_k m_k \sigma_k, \tag{1.3}$$

over the distinct discrete series representations σ_k of G, where $m_k \in \{0, 1, \ldots, \infty\}$ for each k.

In this paper we consider the case when π is the tensor product of two spherical irreducible unitary representations of G. We give an explicit description of π_1 in

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Theorem 4.1 below. We defer the detailed description of π_2 to [1], where somewhat different methods are needed. However, if Γ is a lattice subgroup of G, we completely describe the restriction $\pi_{|\Gamma}$ of π to Γ (see Theorem 5.6 below). This does not require a detailed knowledge of π_2.

The groups G and Γ considered in this paper share with such groups as $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Q}_p)$ the following feature: either exactly one of the complementary series representations appears in the decomposition of the tensor product, or the complementary series does not appear at all (see [6] and [8]).

Our method for finding a description for π_1 is based on a formula for the product of the spherical functions associated with spherical representations (see below). This product formula is interpreted in Section 3 as a product formula for certain orthogonal polynomials, which turns out to be a special case of a product formula for q-ultraspherical polynomials proved by other methods in [7].

We start by recalling the definition of a spherical representation of a group of automorphisms of a homogeneous tree. We shall mostly follow the notation in [3], but it will be convenient to use a different parametrization for these representations.

Let $q \geq 2$ be an integer, let T be a homogeneous tree of degree $q+1$, and let $d(x, y)$ denote the usual distance between vertices x, y of T. For $n \geq 0$ and $x \in T$, the number N_n of vertices y satisfying $d(x, y) = n$ is 1 if $n = 0$, and $(q+1)q^{n-1}$ if $n \geq 1$. Let Ω denote the space of ends of T, i.e., equivalence classes ω of infinite chains in T. If $x \in T$, let $[x, \omega)$ denote the unique infinite chain in the class ω having initial vertex x, and let $\omega_i(x)$ denote the ith vertex of this chain. If $x, y \in T$, let $\Omega_x(y)$ denote the set of $\omega \in \Omega$ such that y is a vertex in $[x, \omega)$. Fixing $x \in T$, there is a totally disconnected compact topology on Ω having the sets $\Omega_x(y)$, $y \in T$, as basis; this topology does not depend on x. We denote by ν_x the regular Borel probability measure on Ω satisfying $\nu_x(\Omega_x(y)) = 1/N_n$ whenever $d(x, y) = n$. For $x, y \in T$, ν_x and ν_y are mutually absolutely continuous, with

$$P(x, y, \omega) := \frac{d\nu_y}{d\nu_x}(\omega) = q^{\delta(x,y,\omega)},$$

where $\delta(x, y, \omega)$ is the unique integer k such that $\omega_i(y) = \omega_{i+k}(x)$ for all large i [3, p. 35].

Note that G acts on Ω in a natural way. For $g \in G$ write $P(g, \omega)$ in place of $P(o, go, \omega)$. Let $\mathcal{K}(\Omega)$ denote the space of locally constant functions on Ω [3, p. 36]. For $z \in \mathbb{C}$, we can define the *spherical representation* π_z of G on $\mathcal{K}(\Omega)$ by

$$\big(\pi_z(g)f\big)(\omega) = P^z(g, \omega) f(g^{-1}\omega).$$

Writing

$$\langle f, g \rangle = \int_\Omega f(\omega)\overline{g(\omega)}\, d\nu_o(\omega), \tag{1.4}$$

the "spherical function" defined on G by

$$g \mapsto \langle \pi_z(g)\mathbf{1}, \mathbf{1} \rangle = \int_\Omega P^z(g, \omega)\, d\nu_o(\omega)$$

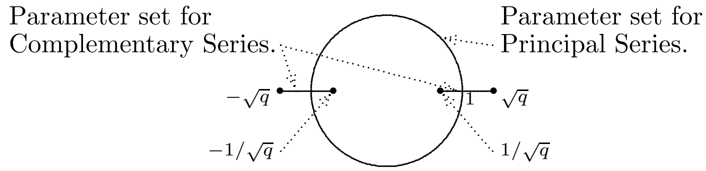

Figure 1

depends only on $n = d(o, go)$. Here $\mathbf{1}$ is the function taking the constant value 1 on Ω.

The explicit formula for the spherical function is neatest if we now change the parametrization of the spherical representations. Write

$$s = q^{\frac{1}{2}-z}, \tag{1.5}$$

and if $s \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$ are so related, we write π^s in place of π_z. Thus

$$\big(\pi^s(g)f\big)(\omega) = \Big(\frac{s}{\sqrt{q}}\Big)^{-\delta(o,go,\omega)} f(g^{-1}\omega) \quad \text{for } f \in \mathcal{K}(\Omega). \tag{1.6}$$

Write

$$\varphi_n(s) = \langle \pi^s(g)\mathbf{1}, \mathbf{1}\rangle = \int_\Omega \big(s/\sqrt{q}\big)^{-\delta(o,go,\omega)}\, d\nu_o(\omega) \quad \text{if } d(o,go) = n. \tag{1.7}$$

Then φ_n is analytic on $\mathbb{C} \setminus \{0\}$, and (see [3, p. 43])

$$\varphi_n(s) = q^{-n/2}\big(c(s)s^n + c(s^{-1})s^{-n}\big), \quad \text{if } s \neq 0, \pm 1, \tag{1.8}$$

where

$$c(s) = \frac{qs - s^{-1}}{(q+1)(s - s^{-1})} \quad \text{for } s \neq 0, \pm 1. \tag{1.9}$$

Also,

$$\varphi_n(s) = \frac{s^n}{(q+1)q^{n/2}}\big((q+1) + (q-1)n\big) \quad \text{if } s = \pm 1.$$

For $s \in \mathbb{T} = \{s \in \mathbb{C} : |s| = 1\}$ (equivalently, $\Re z = 1/2$), $\langle \cdot, \cdot \rangle$ is preserved by each $\pi^s(g)$, and so $\pi^s = \pi_z$ extends to a unitary representation on $L^2(\Omega, \nu_o)$; these representations are called the *principal series* spherical representations. In addition, for s or $-s$ in $(q^{-1/2}, q^{1/2}) \setminus \{1\}$ (equivalently, $1/2 \neq \Re z \in (0,1)$, $\Im z = \pi k/\log q$ for some $k \in \mathbb{Z}$), there is an inner product $\langle \cdot, \cdot \rangle_s$ on $\mathcal{K}(\Omega)$ preserved by each $\pi^s(g)$, and so π^s extends to a unitary representation on the completion H_s of $\mathcal{K}(\Omega)$ with respect to this inner product. If $s = \sqrt{q}$ or $1/\sqrt{q}$ (equivalently, $z = 2k\pi i/\ln q$ or $1 + 2k\pi i/\ln q$ for some $k \in \mathbb{Z}$), then $\varphi_n(s) = 1$ for all n, and we define π^s to be the trivial character of G, instead of using the above definition. Also, if $s = -\sqrt{q}$ or $-1/\sqrt{q}$ (equivalently, $z = (2k+1)\pi i/\ln q$ or $1 + (2k+1)\pi i/\ln q$), then $\varphi_n(s) = (-1)^n$ for all n, and we define π^s to be the character $g \mapsto (-1)^{d(o,go)}$ of G, instead of using the above definition. The representations π^s for s or $-s$ in $[q^{-1/2}, q^{1/2}] \setminus \{1\}$ are called the *complementary series* spherical representations. See Figure 1.

If ds denotes the usual normalized measure on $\mathbb{T}$, let μ be the "Plancherel measure" on $\mathbb{T}$ given by

$$d\mu(s) = \frac{q}{2(q+1)} \frac{1}{|c(s)|^2}\, ds \tag{1.10}$$

(notice that $1/|c(s)|^2$ extends to a continuous function on $\mathbb{T}$). Then it is well known and easy to verify that

$$\int_{\mathbb{T}} \varphi_m(s)\varphi_n(s)\, d\mu(s) = \frac{1}{N_n}\delta_{m,n}. \tag{1.11}$$

2. Formulas for $\varphi_n(s_1)\varphi_n(s_2)$

Proposition 2.1. *If $s_1, s_2, s_3 \in \mathbb{C}\setminus\{0\}$ satisfy*

$$|s_1^{\epsilon_1}s_2^{\epsilon_2}s_3^{\epsilon_3}| < \sqrt{q} \quad \textit{for each } \epsilon = (\epsilon_1,\epsilon_2,\epsilon_3) \in \{+1,-1\}^3, \tag{2.1}$$

then

$$\sum_{n=0}^{\infty} N_n\varphi_n(s_1)\varphi_n(s_2)\varphi_n(s_3) = \frac{q(q-1)}{(q+1)^2}\frac{\prod_{j=1}^3\{(1-\frac{s_j^2}{q})(1-\frac{s_j^{-2}}{q})\}}{\prod_\epsilon(1-\frac{1}{\sqrt{q}}s_1^{\epsilon_1}s_2^{\epsilon_2}s_3^{\epsilon_3})}, \tag{2.2}$$

where the product is over all eight $\epsilon = (\epsilon_1,\epsilon_2,\epsilon_3) \in \{+1,-1\}^3$.

Proof. Assume first that $s_j \neq \pm 1$ for each j, so that $\varphi_n(s_j)$ is given by setting $s = s_j$ in (1.8). For $n \geq 1$, the nth summand on the left in (2.2) is then $(q+1)/q$ times the sum of the eight terms

$$c(s_1^{\epsilon_1})c(s_2^{\epsilon_2})c(s_3^{\epsilon_3})\Big(s_1^{\epsilon_1}s_2^{\epsilon_2}s_3^{\epsilon_3}/\sqrt{q}\Big)^n.$$

Thus, assuming (2.1), the sum on the left in (2.2) is a sum of eight convergent geometric series. So the sum on the left in (2.2) is a sum of eight rational expressions in the s_j's, which after some tedious algebra tidies up to the expression on the right in (2.2).

Suppose that $s_1, s_2, s_3 \in \mathbb{C}$ satisfy (2.1), but that one of the s_j's, say s_3, equals ± 1. Formula (1.7) implies that $|\varphi_n(s)| \leq \varphi_n(|s|)$, and so for arbitrary $s_3 \in \mathbb{T}$ we have $|\varphi_n(s_3)| \leq \varphi_n(1) \leq (n+1)/q^{n/2}$. Hence convergence of the left hand side of (2.2) is uniform with respect to $s_3 \in \mathbb{T}$ for any fixed s_1, s_2 satisfying $|s_1^{\epsilon_1}s_2^{\epsilon_2}| < \sqrt{q}$ for each $(\epsilon_1,\epsilon_2) \in \{+1,-1\}^2$. Hence the sum is a continuous function of s_3. But the right hand side in (2.2) is also a continuous function of s_3 on $\mathbb{T}$ for any fixed such s_1, s_2, and so (2.2) is valid also at $s_3 = \pm 1$ (assuming (2.1), of course). Similarly, (2.2) remains valid if two or three of the s_j's are ± 1, and so the proposition is proved. ∎

Let $K(s_1,s_2,s_3)$ denote the right hand side of (2.2) whenever the denominator is nonzero. Notice that $K(s_1,s_2,s_3)$ is symmetric in s_1, s_2 and s_3, and is unchanged if any s_j is replaced by s_j^{-1}. Notice also that $K(s_1,s_2,s_3) > 0$ for all $s_3 \in \mathbb{T}$ when s_1 and s_2 are parameters corresponding to either the principal or complementary series, except that when s_1 or $s_2 \in \{\pm\sqrt{q}, \pm 1/\sqrt{q}\}$, $K(s_1,s_2,s_3) = 0$ for all $s_3 \in \mathbb{T}$.

If $|s_1^{\epsilon_1}s_2^{\epsilon_2}| \neq \sqrt{q}$ for each $\epsilon = (\epsilon_1,\epsilon_2) \in \{+1,-1\}^2$, then $K(s_1,s_2,s_3)$ is a continuous function of s_3 on $\mathbb{T}$, and is therefore integrable with respect to μ.

Proposition 2.2. *Consider $s_1, s_2 \in \mathbb{C}\setminus\{0\}$ satisfying*

$$|s_1^{\epsilon_1}s_2^{\epsilon_2}| < \sqrt{q} \quad \textit{for each } \epsilon = (\epsilon_1,\epsilon_2) \in \{+1,-1\}^2. \tag{2.3}$$

Then the following "multiplication formula" holds for each $m \in \mathbb{N}$:

$$\varphi_m(s_1)\varphi_m(s_2) = \int_{\mathbb{T}} K(s_1,s_2,s_3)\varphi_m(s_3)\, d\mu(s_3). \tag{2.4}$$

Proof. Apply (2.2) to (s_1, s_2, s_3) for arbitrary $s_3 \in \mathbb{T}$. We multiply both sides of (2.2) by $\varphi_m(s_3)$, integrate with respect to $d\mu(s_3)$, and get (2.4). This is valid because the convergence of the sum on the left in (2.2) is uniform with respect to $s_3 \in \mathbb{T}$, as we saw in the proof of Proposition 2.1 above. □

We next show that (2.4) holds in another case.

Proposition 2.3. *Suppose that $s_1, s_2 \in \mathbb{C}\setminus\{0\}$, and that one of the $s_1^{\epsilon_1}s_2^{\epsilon_2}$'s, where $(\epsilon_1, \epsilon_2) \in \{+1, -1\}^2$, equals $\pm\sqrt{q}$, and the other three have modulus less than $\sqrt{q}$. Then $K(s_1, s_2, s_3)$ is integrable with respect to $d\mu(s_3)$, and (2.4) holds.*

Proof. By the symmetry properties of $K(s_1, s_2, s_3)$, and because $\varphi_n(s) = \varphi_n(s^{-1})$, we may suppose that $s_1 s_2 = \sqrt{q}$ or $-\sqrt{q}$. For $s_3 \in \mathbb{T}$, we see from (1.9) and (2.2) that

$$\frac{q}{2(q+1)}\frac{K(s_1, s_2, s_3)}{|c(s_3)|^2} = \frac{(q-1)}{2(q+1)}\frac{|s_3^2 - 1|^2\prod_{j=1}^2\left\{\left(1 - \frac{s_j^2}{q}\right)\left(1 - \frac{s_j^{-2}}{q}\right)\right\}}{\prod_\epsilon(1 - \frac{1}{\sqrt{q}}s_1^{\epsilon_1}s_2^{\epsilon_2}s_3^{\epsilon_3})}, \tag{2.5}$$

where the product in the denominator is over all eight $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3$. The product of two of the factors in the denominator in (2.5) is

$$(1 - s_3)(1 - s_3^{-1}) \quad \text{or} \quad (1 + s_3)(1 + s_3^{-1}). \tag{2.6}$$

In either case, the factor $|s_3^2 - 1|^2$ in the numerator on the right in (2.5) allows us to cancel the two factors in (2.6). As the other six factors in the denominator are bounded away from zero, $K(s_1, s_2, s_3)$ is integrable with respect to $d\mu(s_3)$. Now let $0 < r < 1$. Then rs_1, s_2 satisfy (2.3) if r is close to 1. Elementary calculus shows that $|1 - e^{i\theta}|/|1 - re^{i\theta}| \leq 2/(1 + r) < 2$ for $\theta \in \mathbb{R}$ and $1 \neq r > 0$. Using this, we see that for $r < 1$ close to 1, $|K(rs_1, s_2, s_3)| \leq M|K(s_1, s_2, s_3)|$ for a constant M. The Dominated Convergence Theorem shows that

$$\int_{\mathbb{T}} K(rs_1, s_2, s_3)\varphi_m(s_3)\, d\mu(s_3) \to \int_{\mathbb{T}} K(s_1, s_2, s_3)\varphi_m(s_3)\, d\mu(s_3) \quad \text{as } r \to 1. \tag{2.7}$$

Certainly the left hand side of (2.4) is continuous in s_1 and s_2, and so Proposition 2.2 and (2.7) show that (2.4) holds. □

One can show that, apart from under the conditions of Propositions 2.2 and 2.3, there is only one other case when (2.4) holds: when one of the $s_1^{\epsilon_1}s_2^{\epsilon_2}$'s equals $\sqrt{q}$, and another equals $-\sqrt{q}$. We omit the proof, as this cannot arise when s_1, s_2 are parameters corresponding to the principal or complementary series.

Corollary 2.4. *Suppose that either*

(i) *s_1, s_2 are parameters corresponding to the principal series, or that*
(ii) *s_1 corresponds to the principal series and $s_2 \notin \{\pm\sqrt{q}, \pm 1/\sqrt{q}\}$ corresponds to the complementary series, or vice versa, or*
(iii) *$s_1, s_2 \notin \{\pm\sqrt{q}, \pm 1/\sqrt{q}\}$ correspond to the complementary series, and $|s_1^{\epsilon_1}s_2^{\epsilon_2}| \leq \sqrt{q}$ for each $(\epsilon_1, \epsilon_2) \in \{-1, +1\}^2$.*

Then (2.4) holds, and $K(s_1, s_2, s_3) > 0$ for all $s_3 \in \mathbb{T}$.

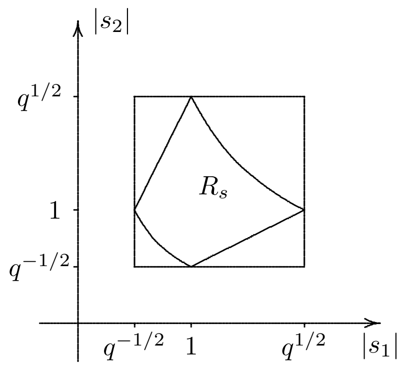

Figure 2

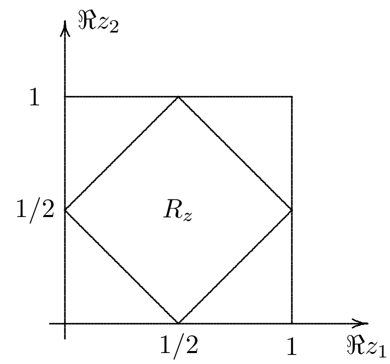

Figure 3

We remark that when, say, $s_2 \in \{\sqrt{q}, 1/\sqrt{q}\}$, then $\varphi_m(s_2) = 1$ for all m, and there is a simple multiplication formula: $\varphi_m(s_1)\varphi_m(s_2) = \varphi_m(s_1)$. When $s_2 \in \{-\sqrt{q}, -1/\sqrt{q}\}$, then $\varphi_m(s_2) = (-1)^m$ for all m, and we have $\varphi_m(s_1)\varphi_m(s_2) = \varphi_m(-s_1)$.

Notice that part (iii) of Corollary 2.4, and the last remark, do not cover all cases when s_1, s_2 correspond to the complementary series. The region covered is indicated by R_s in Figure 2. In terms of parameters z_j satisfying $s_j = q^{1/2-z_j}$, the region covered is the tilted square R_z indicated in Figure 3.

We now consider modifications of (2.4) which hold when $|s_1^{\epsilon_1} s_2^{\epsilon_2}| > \sqrt{q}$ for some (ϵ_1, ϵ_2). When s_1, s_2 correspond to the complementary series, this will cover the cases not covered above. We may assume that $|s_1 s_2|$ is the largest of the $|s_1^{\epsilon_1} s_2^{\epsilon_2}|$.

Proposition 2.5. *Suppose that* $s_1, s_2 \in \mathbb{C} \setminus \{0\}$ *satisfy*

$$|s_1 s_2| > \sqrt{q} > \max\{|s_1 s_2^{-1}|, |s_1^{-1} s_2|, |s_1^{-1} s_2^{-1}|\}. \tag{2.8}$$

Then for $A = c(s_1)c(s_2)/c(s_1 s_2/\sqrt{q})$, *we have*

$$\varphi_m(s_1)\varphi_m(s_2) = A\varphi_m(s_1 s_2/\sqrt{q}) + \int_{\mathbb{T}} K(s_1, s_2, s_3)\varphi_m(s_3)\, d\mu(s_3). \tag{2.9}$$

Proof. As is implicit in the proof of Proposition 2.1, we have

$$K(s_1, s_2, s_3) = 1 + \frac{q+1}{q} \sum_{\epsilon} \frac{c(s_1^{\epsilon_1})c(s_2^{\epsilon_2})c(s_3^{\epsilon_3}) s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3}}{\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3}},$$

whenever $s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3} \neq \sqrt{q}$ for all ϵ (the sum is over all eight $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3$). Using (1.10) and (1.11), and $\overline{c(s)} = c(s^{-1})$ for $s \in \mathbb{T}$, and writing $I_m = \int_{\mathbb{T}} K(s_1, s_2, s_3)\varphi_m(s_3)\, d\mu(s_3)$, we see that

$$I_m = \delta_{m,0} + \frac{1}{2} \sum_{\epsilon} c(s_1^{\epsilon_1})c(s_2^{\epsilon_2}) s_1^{\epsilon_1} s_2^{\epsilon_2} \int_{\mathbb{T}} \frac{s_3^{\epsilon_3}\varphi_m(s_3)}{(\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3}) c(s_3^{-\epsilon_3})}\, ds_3. \tag{2.10}$$

Now using $\int_{\mathbb{T}} f(s)\, ds = \int_{\mathbb{T}} f(s^{-1})\, ds$ and $\varphi_m(s) = \varphi_m(s^{-1})$, we see that for given $\epsilon_1, \epsilon_2 \in \{+1, -1\}$, the integral in (2.10) is the same for $\epsilon_3 = +1$ as for $\epsilon_3 = -1$. So

$$I_m = \delta_{m,0} + \sum_{\epsilon_1, \epsilon_2 = \pm 1} c(s_1^{\epsilon_1})c(s_2^{\epsilon_2}) s_1^{\epsilon_1} s_2^{\epsilon_2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}\varphi_m(e^{i\theta})}{(\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} e^{i\theta}) c(e^{-i\theta})}\, d\theta \right\}.$$

The expression in the braces equals

$$\frac{1}{2\pi i}\oint_{|w|=1}\frac{\varphi_m(w)}{(\sqrt{q}-s_1^{\epsilon_1}s_2^{\epsilon_2}w)c(w^{-1})}\,dw, \tag{2.11}$$

which we can evaluate using the residue theorem. Assuming $|s_1^{\epsilon_1}s_2^{\epsilon_2}|<\sqrt{q}$, the singularity $\sqrt{q}/(s_1^{\epsilon_1}s_2^{\epsilon_2})$ of the integrand in (2.11) is outside the unit circle. Next write $\varphi_m(w)=q^{-m/2}\big(c(w)w^m+c(w^{-1})w^{-m}\big)$. Now

$$\oint_{|w|=1}\frac{c(w)w^m}{(\sqrt{q}-s_1^{\epsilon_1}s_2^{\epsilon_2}w)c(w^{-1})}\,dw=\oint_{|w|=1}\frac{w^m}{\sqrt{q}-s_1^{\epsilon_1}s_2^{\epsilon_2}w}\frac{qw^2-1}{w^2-q}\,dw=0,$$

because the integrand is analytic on and inside the unit circle. On the other hand,

$$\oint_{|w|=1}\frac{c(w^{-1})w^{-m}}{(\sqrt{q}-s_1^{\epsilon_1}s_2^{\epsilon_2}w)c(w^{-1})}\,dw=\oint_{|w|=1}\frac{1}{\sqrt{q}}\left\{\sum_{k=0}^{\infty}\Big(\frac{s_1^{\epsilon_1}s_2^{\epsilon_2}w}{\sqrt{q}}\Big)^k\right\}\frac{1}{w^m}\,dw,$$

which equals $2\pi i(s_1^{\epsilon_1}s_2^{\epsilon_2})^{m-1}/q^{m/2}$ if $m\geq 1$ and equals 0 if $m=0$.

If, however, $|s_1^{\epsilon_1}s_2^{\epsilon_2}|>\sqrt{q}$, then the simple pole $\sqrt{q}/(s_1^{\epsilon_1}s_2^{\epsilon_2})$ of the integrand in (2.11) is inside the unit circle, and the residue is

$$-\frac{\varphi_m(\sqrt{q}/s_1^{\epsilon_1}s_2^{\epsilon_2})}{s_1^{\epsilon_1}s_2^{\epsilon_2}c(s_1^{\epsilon_1}s_2^{\epsilon_2}/\sqrt{q})}.$$

Using $\varphi_m(s)=\varphi_m(s^{-1})$, the result is now clear. □

When s_1,s_2 are parameters corresponding to the principal or complementary series, it cannot happen that $|s_1^{\epsilon_1}s_2^{\epsilon_2}|>\sqrt{q}$ holds for two (ϵ_1,ϵ_2)'s. However, we mention for the sake of completeness that if $s_2\neq\pm1$ and $|s_1s_2|\geq|s_1s_2^{-1}|>\sqrt{q}$, then (2.9) holds, provided an extra term $B\varphi_m(s_1s_2^{-1}/\sqrt{q})$, where $B=c(s_1)c(s_2^{-1})/c(s_1s_2^{-1}/\sqrt{q})$, is added on the right.

3. Connection with orthogonal polynomials

As is well known, and clear from the relation

$$\varphi_1(s)\varphi_n(s)=\frac{q\varphi_{n+1}(s)+\varphi_{n-1}(s)}{q+1},$$

each function $\varphi_n(s)$ is a polynomial $p_n(t)$ in

$$t=\varphi_1(s)=\frac{\sqrt{q}}{q+1}\Big(s+\frac{1}{s}\Big).$$

Note that φ_1 maps $\mathbb{T}$ onto $I=[-2\sqrt{q}/(q+1),2\sqrt{q}/(q+1)]$. The image $\tilde{\mu}$ of the Plancherel measure μ under this map has density $\sqrt{4q-(q+1)^2t^2}/(2\pi(1-t^2))$ with respect to Lebesgue measure on I. Formula (1.11) becomes

$$\int_I p_m(t)p_n(t)\,d\tilde{\mu}(t)=\frac{1}{N_n}\delta_{m,n}.$$

Setting $t_j=\varphi_1(s_j)$ for $j=1,2,3$, we can express $K(s_1,s_2,s_3)$ in terms of t_1,t_2,t_3: $K(s_1,s_2,s_3)=\tilde{K}(t_1,t_2,t_3)$ for

$$\tilde{K}(t_1,t_2,t_3)=\frac{(q-1)(1-t_1^2)(1-t_2^2)(1-t_3^2)}{D},$$

where

$$D = (q+1)^2 t_1^2 t_2^2 t_3^2 - (q+1)^2 (t_1^3 t_2 t_3 + t_1 t_2^3 t_3 + t_1 t_2 t_3^3) + q(t_1^4 + t_2^4 + t_3^4) + (q^2+1)(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2) - (q^2 - 6q + 1) t_1 t_2 t_3 - 2q(t_1^2 + t_2^2 + t_3^2) + q.$$

Thus, as a special case of Proposition 2.2, we know that $\tilde{K}(t_1, t_2, t_3) > 0$ for $t_1, t_2, t_3 \in I$, and that, for any $t_1, t_2 \in I$ and $m \in \mathbb{N}$,

$$p_m(t_1) p_m(t_2) = \int_I \tilde{K}(t_1, t_2, t_3) p_m(t_3)\, d\tilde{\mu}(t_3). \tag{3.1}$$

This formula is a special case of the product formula for q-ultraspherical polynomials found by Rahman and Verma [7, (1.20)]. The "q" here is not our q. Let us temporarily write Q in place of our q. Take the a and q of [7, (1.20)] to be $1/\sqrt{Q}$ and 0, respectively. Then the polynomial $p_n(x; a, a\sqrt{q}, -a, -a\sqrt{q})$ of [7] equals $p_n(2\sqrt{Q}\, x/(Q+1))$, and [7, (1.20)] specializes to (3.1).

4. Decomposing $\pi^{s_1} \otimes \pi^{s_2}$

Let $\mathcal{K}(\Omega)$ and $\mathcal{K}(\Omega \times \Omega)$ denote the space of locally constant functions on Ω and $\Omega \times \Omega$, respectively. We can identify the (algebraic) tensor product $\mathcal{K}(\Omega) \otimes \mathcal{K}(\Omega)$ with $\mathcal{K}(\Omega \times \Omega)$, the identification being induced by the assignment $f_1 \otimes f_2 \mapsto F$, where $F(\omega_1, \omega_2) = f_1(\omega_1) f_2(\omega_2)$. Thus if $s_1, s_2 \in \mathbb{C} \setminus \{0\}$, we can consider the tensor product representation $\pi = \pi^{s_1} \otimes \pi^{s_2}$ of $G = \mathrm{Aut}(T)$ as having representation space $\mathcal{K}(\Omega \times \Omega)$, and given for $F \in \mathcal{K}(\Omega \times \Omega)$ by

$$\big(\pi(g)F\big)(\omega_1, \omega_2) = \Big(\frac{s_1}{\sqrt{q}}\Big)^{-\delta(o, go, \omega_1)} \Big(\frac{s_2}{\sqrt{q}}\Big)^{-\delta(o, go, \omega_2)} F(g^{-1}\omega_1, g^{-1}\omega_2).$$

We are mainly concerned with the case when s_1 and s_2 correspond to the principal or complementary series. Then there are inner products $\langle \cdot, \cdot \rangle_{s_i}$ on $\mathcal{K}(\Omega)$ so that π^{s_i} extends to a unitary representation of G on the completion H_{s_i} of $\mathcal{K}(\Omega)$ with respect to $\langle \cdot, \cdot \rangle_{s_i}$. Also, π extends to a unitary representation on the completion H_{s_1, s_2} of $\mathcal{K}(\Omega \times \Omega)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{s_1, s_2}$ determined by

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{s_1, s_2} = \langle f_1, g_1 \rangle_{s_1} \langle f_2, g_2 \rangle_{s_2} \quad (\text{for } f_1, f_2, g_1, g_2 \in \mathcal{K}(\Omega)).$$

Our aim is to decompose π into irreducible representations. For this purpose, we may suppose that $s_1, s_2 \neq \pm\sqrt{q}, \pm 1/\sqrt{q}$. For π is equivalent to π^{s_1} if $s_2 = \sqrt{q}$ or $s_2 = 1/\sqrt{q}$, while π is equivalent to π^{-s_1} if $s_2 = -\sqrt{q}$ or $s_2 = -1/\sqrt{q}$ (see the definition in Section 1 of π^{s_2} in these special cases).

Let H_1 and H_2 be as in (1.1), and let π_1 and π_2 be the restrictions of π to these invariant subspaces.

Theorem 4.1. *If s_1 and s_2 are parameters corresponding to the principal or complementary series, with $s_1, s_2 \notin \{\pm\sqrt{q}, \pm 1/\sqrt{q}\}$, then H_1 equals the closure in H_{s_1, s_2} of the linear span of $\{\pi(g)(\mathbf{1} \otimes \mathbf{1}) : g \in G\}$. Under the conditions of Corollary 2.4 above, we have*

$$\pi_1 \cong \int_{\mathbb{T}}^{\oplus} \pi^s\, K(s_1, s_2, s) d\mu(s) \cong \int_{\mathbb{T}}^{\oplus} \pi^s\, ds,$$

while under the hypothesis of (2.8), letting $s_3 = s_1 s_2/\sqrt{q}$, we have

$$\pi_1 \cong \pi^{s_3} \oplus \int_{\mathbb{T}}^{\oplus} \pi^s\, K(s_1, s_2, s) d\mu(s) \cong \pi^{s_3} \oplus \int_{\mathbb{T}}^{\oplus} \pi^s\, ds.$$

In both cases, $K(s_1, s_2, s) > 0$ for all $s \in \mathbb{T}$.

Proof. To show that H_1 is equal to the closure in H_{s_1,s_2} of the linear span of $\{\pi(g)(\mathbf{1}\otimes\mathbf{1}) : g \in G\}$, we must show that any K-invariant $\xi \in H_{s_1,s_2}$ is in this closure. The orthogonal projection P of H_{s_1,s_2} onto the space of K-invariant vectors is $\int_K \pi(k)\, dk$, where the integral is with respect to normalized Haar measure on K, and it is clear that P leaves $\mathcal{K}(\Omega\times\Omega)$ invariant. So it is enough to check that each K-invariant $F \in \mathcal{K}(\Omega\times\Omega)$ belongs to the linear span of $\{\pi(g)(\mathbf{1}\otimes\mathbf{1}) : g \in G\}$. This is proved in Lemma 4.2 below.

To obtain the integral decomposition, we now form the direct integral $\pi_{\text{int}} = \int_{\mathbb{T}}^{\oplus} \pi^s K(s_1, s_2, s)\, d\mu(s)$ of the π^s's, with representation space H_{int}, say. For each parameter s, let $v_s \in H_s$ be a unit cyclic vector for π^s such that $\langle \pi^s(g)v_s, v_s\rangle = \varphi_m(s)$ whenever $d(o, go) = m$. Then $v = \int_{\mathbb{T}}^{\oplus} v_s K(s_1, s_2, s)\, d\mu(s)$ is, under the conditions of Corollary 2.4, a unit vector for π_{int} which is cyclic (cf. [5, Theorem P.2 (p. 97) and Theorem 2.9 (p. 108)]) and satisfies

$$\langle \pi_{\text{int}}(g)v, v\rangle = \int_{\mathbb{T}} \varphi_m(s)K(s_1, s_2, s)\, d\mu(s) = \varphi_m(s_1)\varphi_m(s_2)$$

whenever $d(o, go) = m$. Hence π_1 is equivalent to π_{int}. Under the conditions of (2.8), let $B = \int_{\mathbb{T}} K(s_1, s_2, s)\, d\mu(s) = 1 - A$. Then v above has norm $\sqrt{B}$, and $u = (\sqrt{A}v_{s_3}, v)$ is a unit vector in $H_{s_3} \oplus H_{\text{int}}$, and the positive definite function obtained from u and $\pi^{s_3} \oplus \pi_{\text{int}}$ is given on the right hand side of (2.9). Moreover, π^{s_3} and π_{int} are disjoint [5, p. 16], because π_{int} is the direct integral of representations weakly contained in the left regular representation, and hence itself weakly contained therein, whereas π^{s_3} is irreducible and not weakly contained in the left regular representation, being from the complementary spherical series. Hence u above is cyclic for $H_{s_3} \oplus H_{\text{int}}$, by [5, p. 52, Corollary]. Hence π_1 is equivalent to $\pi^{s_3} \oplus \pi_{\text{int}}$.

The second equivalences hold by [5, p. 87, Lemma], as $s \mapsto K(s_1, s_2, s)/|c(s)|^2$ is integrable with respect to ds, and strictly positive (except at $s = \pm i$). □

Lemma 4.2. *Suppose that* $s_1, s_2 \in \mathbb{C}\setminus\{0, \pm\sqrt{q}\}$, *and form* $\pi = \pi^{s_1} \otimes \pi^{s_2}$. *Suppose that* $F \in \mathcal{K}(\Omega\times\Omega)$ *is* K*-invariant. Then* F *is in the linear span of* $\{\pi(g)(\mathbf{1}\otimes\mathbf{1}) : g \in G\}$.

Proof. For $k \in K$ and $F \in \mathcal{K}(\Omega\times\Omega)$, $\big(\pi(k)F\big)(\omega_1, \omega_2) = F(k^{-1}\omega_1, k^{-1}\omega_2)$. So F is K-invariant if and only if (i) $F(\omega, \omega)$ is independent of $\omega \in \Omega$, and (ii) for $\omega_1 \neq \omega_2$, $F(\omega_1, \omega_2)$ depends only on $d(\omega_1 \wedge \omega_2, o)$. Here $\omega_1 \wedge \omega_2$ denotes the confluent of ω_1 and ω_2, i.e., the last vertex common to $[o, \omega_1)$ and $[o, \omega_2)$. For $n \in \mathbb{N}$, define $F_n \in \mathcal{K}(\Omega\times\Omega)$ by setting

$$F_n(\omega_1, \omega_2) = \begin{cases} 1 & \text{if } \omega_1 \neq \omega_2 \text{ and } d(\omega_1 \wedge \omega_2, o) = n, \\ 0 & \text{otherwise.} \end{cases}$$

So the linear span of $\mathbf{1}\otimes\mathbf{1}$ and the functions F_n consists of the K-invariant functions $F \in \mathcal{K}(\Omega\times\Omega)$. To prove the lemma, it is sufficient to show that each function F_n is in the linear span of $\{\pi(g)(\mathbf{1}\otimes\mathbf{1}) : g \in G\}$.

For $v \in T$, let $\mathbf{1}_v \in \mathcal{K}(\Omega)$ denote the indicator function of $\Omega(v) = \Omega_o(v)$. Let us write $\mathcal{C}_n$ for the set of vertices $v \in T$ such that $d(v, o) = n$. For $v \in T$, choose $g_v \in G$ such that $g_v o = v$. Suppose now that $v \in \mathcal{C}_1$. For $s \in \mathbb{C}\setminus\{0\}$ and $\omega \in \Omega$,

$\big(\pi^s(g_v)\mathbf{1}\big)(\omega)$ equals $s/\sqrt{q}$ or $(s/\sqrt{q})^{-1}$ according as $\omega \notin \Omega(v)$ or $\omega \in \Omega(v)$. Thus

$$\big(\pi^s(g_v)\mathbf{1}\big)(\omega) = \frac{s}{\sqrt{q}}\mathbf{1}(\omega) + \Big(\frac{\sqrt{q}}{s} - \frac{s}{\sqrt{q}}\Big)\mathbf{1}_v(\omega).$$

Suppose now $s_1, s_2 \in \mathbb{C} \setminus \{0\}$, $\omega_1, \omega_2 \in \Omega$ and $v \in \mathcal{C}_1$. Then

$$\begin{aligned}&\big((\pi^{s_1}(g_v)\mathbf{1}) \otimes (\pi^{s_1}(g_v)\mathbf{1})\big)(\omega_1,\omega_2)\\ &\quad= \Big\{\frac{s_1}{\sqrt{q}}\mathbf{1}(\omega_1) + \Big(\frac{\sqrt{q}}{s_1} - \frac{s_1}{\sqrt{q}}\Big)\mathbf{1}_v(\omega_1)\Big\} \times \Big\{\frac{s_2}{\sqrt{q}}\mathbf{1}(\omega_2) + \Big(\frac{\sqrt{q}}{s_2} - \frac{s_2}{\sqrt{q}}\Big)\mathbf{1}_v(\omega_2)\Big\}.\end{aligned}$$

If we now sum this identity over the $q+1$ v's in $\mathcal{C}_1$, and use $\sum_{v\in\mathcal{C}_1}\mathbf{1}_v = \mathbf{1}$, we get

$$\begin{aligned}\sum_{v\in\mathcal{C}_1}\big(\pi(g_v)(\mathbf{1}\otimes\mathbf{1})\big)(\omega_1,\omega_2) &= \Big(\frac{q-1}{q}s_1s_2 + \frac{s_1}{s_2} + \frac{s_2}{s_1}\Big)(\mathbf{1}\otimes\mathbf{1})(\omega_1,\omega_2)\\ &\qquad + \frac{(q-s_1^2)(q-s_2^2)}{qs_1s_2}\sum_{v\in\mathcal{C}_1}\mathbf{1}_v(\omega_1)\mathbf{1}_v(\omega_2).\end{aligned}$$

Now the sum $\sum_{v\in\mathcal{C}_1}\mathbf{1}_v(\omega_1)\mathbf{1}_v(\omega_2)$ equals 1 if and only if the geodesics $[o,\omega_1)$ and $[o,\omega_1)$ have a vertex in common other than o, and is zero otherwise. It therefore equals $(\mathbf{1}\otimes\mathbf{1})(\omega_1,\omega_2) - F_0(\omega_1,\omega_2)$. Substituting this into the last expression, we get

$$\begin{aligned}&\sum_{v\in\mathcal{C}_1}\big(\pi(g_v)(\mathbf{1}\otimes\mathbf{1})\big)(\omega_1,\omega_2)\\ &\qquad= \Big(s_1s_2 + \frac{q}{s_1s_2}\Big)(\mathbf{1}\otimes\mathbf{1})(\omega_1,\omega_2) - \frac{(q-s_1^2)(q-s_2^2)}{qs_1s_2}F_0(\omega_1,\omega_2).\end{aligned}$$

As $(q-s_1^2)(q-s_2^2) \neq 0$ by hypothesis, we see that F_0 is in the linear span of $\{\pi(g)(\mathbf{1}\otimes\mathbf{1}) : g \in G\}$.

Now suppose that $n \geq 2$, and that we have shown that $F_0, \dots, F_{n-2}$ are in the linear span of $\{\pi(g)(\mathbf{1}\otimes\mathbf{1}) : g \in G\}$. Suppose that $v \in \mathcal{C}_n$. Let $v_0 = o, v_1, \dots, v_n = v$ be the geodesic from o to v. Then $(\pi^s(g_v)\mathbf{1})(\omega) = (s/\sqrt{q})^{n-2j}$ if v_j is the last vertex common to $[o,\omega)$ and $[v,\omega)$. Thus

$$\begin{aligned}\pi^s(g_v)\mathbf{1} &= \Big(\frac{s}{\sqrt{q}}\Big)^n\mathbf{1}_{\Omega\setminus\Omega(v_1)} + \sum_{j=1}^{n-1}\Big(\frac{s}{\sqrt{q}}\Big)^{n-2j}\mathbf{1}_{\Omega(v_j)\setminus\Omega(v_{j+1})} + \Big(\frac{s}{\sqrt{q}}\Big)^{-n}\mathbf{1}_{\Omega(v_n)}\\ &= \sum_{j=0}^{n} c_j(s)\mathbf{1}_{v_j}, \quad \text{say.}\end{aligned}$$

Note that $c_n(s) = \big(\frac{s}{\sqrt{q}}\big)^{-n}\big(1 - \frac{s^2}{q}\big) \neq 0$ if $s \neq \pm\sqrt{q}$. Hence for $\omega_1, \omega_2 \in \Omega$,

$$\begin{aligned}\big(\pi(g_v)(\mathbf{1}\otimes\mathbf{1})\big)(\omega_1,\omega_2) &= \big(\pi^{s_1}(g_v)\mathbf{1}\big)(\omega_1)\big(\pi^{s_2}(g_v)\mathbf{1}\big)(\omega_2)\\ &= \sum_{j,k=0}^{n} c_j(s_1)c_k(s_2)\mathbf{1}_{v_j}(\omega_1)\mathbf{1}_{v_k}(\omega_2).\end{aligned}$$

We now wish to sum this last equation over all $v \in \mathcal{C}_n$, and so need to calculate

$$\sum_{v\in\mathcal{C}_n}\mathbf{1}_{v_j}(\omega_1)\mathbf{1}_{v_k}(\omega_2).$$

This clearly equals

$$\sum_{u\in\mathcal{C}_j}\sum_{u'\in\mathcal{C}_k} N(u,u')\mathbf{1}_u(\omega_1)\mathbf{1}_{u'}(\omega_2), \tag{4.1}$$

where for $u\in\mathcal{C}_j$ and $u'\in\mathcal{C}_k$, $N(u,u')=\sharp\{v\in\mathcal{C}_n : u,u'\in[o,v]\}$.

To calculate $N(u,u')$, first suppose that $j\le k$. Then $N(u,u')=0$ if $u\notin[o,u']$, while if $u\in[o,u']$, then $N(u,u')$ is a number $K_{n,k}$ independent of $u'\in\mathcal{C}_k$. Hence for each $u'\in\mathcal{C}_k$, there is only one $u\in\mathcal{C}_j$ such that $N(u,u')\ne 0$, namely the jth vertex u'_j on the geodesic $[o,u']$. Moreover, given ω_2, there is only one $u'\in\mathcal{C}_k$ such that $\mathbf{1}_{u'}(\omega_2)\ne 0$, namely $u'=(\omega_2)_k$, the kth vertex on the geodesic $[o,\omega_2)$. Thus the sum (4.1) equals

$$K_{n,k}\mathbf{1}_{u'_j}(\omega_1)=\begin{cases}K_{n,k} & \text{if } (\omega_1)_j=(\omega_2)_j,\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\sum_{v\in\mathcal{C}_n}\big(\pi(g_v)(\mathbf{1}\otimes\mathbf{1})\big)(\omega_1,\omega_2)=\sum_{j,k=0}^{n} c_j(s_1)c_k(s_2)K_{n,j\vee k}\,\delta_{(\omega_1)_{j\wedge k},(\omega_2)_{j\wedge k}},$$

where $j\vee k=\max\{j,k\}$ and $j\wedge k=\min\{j,k\}$. Now

$$\delta_{(\omega_1)_{j\wedge k},(\omega_2)_{j\wedge k}}=1\Leftrightarrow d(\omega_1\wedge\omega_2,o)\ge j\wedge k\Leftrightarrow\Big(\mathbf{1}\otimes\mathbf{1}-\sum_{\ell=0}^{j\wedge k-1}F_\ell\Big)(\omega_1,\omega_2)=1.$$

It follows that $\sum_{v\in\mathcal{C}_n}\pi(g_v)(\mathbf{1}\otimes\mathbf{1})$ is a linear combination of $\mathbf{1}\otimes\mathbf{1}$ and $F_0,\dots,F_{n-1}$. The coefficient of F_{n-1} equals $c_n(s_1)c_n(s_2)K_{n,n}=c_n(s_1)c_n(s_2)\ne 0$. By the induction hypothesis we see that F_{n-1} is in the linear span of $\{\pi(g)(\mathbf{1}\otimes\mathbf{1}) : g\in G\}$. □

5. Restriction to a lattice subgroup

Throughout this section, let Γ denote a lattice subgroup of G, i.e., a discrete subgroup of G such that G/Γ carries a finite G-invariant measure. The discreteness implies that $\Gamma_o=\Gamma\cap K=\{\gamma\in\Gamma:\gamma o=o\}$ is finite. Examples of lattice subgroups Γ include all discrete co-compact subgroups, which are the discrete subgroups having only a finite number of orbits on T. Very special examples of these are the subgroups which act transitively and faithfully (so that $\Gamma_o=\{1\}$), which are described in [3, Thm. I.6.3 and Appendix, Prop. 5.5]. Here 1 denotes the identity automorphism of T. For examples of lattice subgroups that are not co-compact, see [9, p. 88] and [4]. Apart from discreteness, the only property of lattice subgroups we need is contained in the next lemma.

We shall use the notation $T_o(x)=\{y\in T: x\in[o,y]\}$, by analogy with the notation $\Omega_o(x)$, and let $\Gamma_x=\{\gamma\in\Gamma:\gamma x=x\}$ for $x\in T$.

Lemma 5.1. *Assume that Γ is a lattice subgroup of G. Then for any $\omega\in\Omega$ there is a sequence (γ_n) in Γ such that $\gamma_n o\to\omega$.*

Proof. 1. We first show that, given $\omega\in\Omega$, there is a sequence (x_n) in T and there is a sequence (γ_n) of distinct elements of Γ such that $x_n\to\omega$ and $\gamma_n x_n\to\omega$. Let $(g_\alpha)_{\alpha\in A}$ be a set of double coset representatives for $\Gamma\backslash G/K$. Let $v_\alpha=g_\alpha o$, and let $\Gamma_\alpha=\Gamma_{v_\alpha}=\Gamma\cap g_\alpha K g_\alpha^{-1}$. The fact that Γ is a lattice in G is equivalent to $\sum_\alpha 1/|\Gamma_\alpha|<\infty$ (see [9, p. 84]). In particular, either A is finite, or the numbers $|\Gamma_\alpha|$ are unbounded.

Let $[o,\omega) = (y_0 = o, y_1, \dots)$. We inductively choose $x_n \in T_o(y_n)$ and $\gamma_n \in \Gamma$ so that $\gamma_n x_n \in T_o(y_n)$ and $\gamma_0, \dots, \gamma_n$ are distinct. We start by setting $x_0 = o$ and $\gamma_0 = 1$. Suppose that $n \geq 1$ and that $x_0, \dots, x_{n-1}$ and $\gamma_0, \dots, \gamma_{n-1}$ have been chosen. Now $\gamma v_\alpha \in T_o(y_n)$ for infinitely many pairs $(\gamma, \alpha) \in \Gamma \times A$. If there is an α so that $\gamma v_\alpha \in T_o(y_n)$ for infinitely many $\gamma \in \Gamma$, we can choose $\gamma, \gamma' \in \Gamma$ so that $\gamma v_\alpha, \gamma' v_\alpha \in T_o(y_n)$ and so that $\gamma'\gamma^{-1} \notin \{\gamma_0, \dots, \gamma_{n-1}\}$. Let $x_n = \gamma v_\alpha$ and $\gamma_n = \gamma'\gamma^{-1}$. If, for each $\alpha \in A$, $\gamma v_\alpha \in T_o(y_n)$ holds for only finitely many $\gamma \in \Gamma$, choose an α so that $|\Gamma_\alpha| > n$ and so that $\gamma v_\alpha \in T_o(y_n)$ for some $\gamma \in \Gamma$. Fix such a γ, and let $x_n = \gamma v_\alpha$. As $|\Gamma_{x_n}| = |\gamma \Gamma_\alpha \gamma^{-1}| = |\Gamma_\alpha| > n$, we can choose γ_n to be any element of $\Gamma_{x_n} \setminus \{\gamma_0, \dots, \gamma_{n-1}\}$.

2. Let $\omega \in \Omega$ be given, and let (x_n) and (γ_n) be as in step 1. We show that a subsequence of $(\gamma_n o)$ or of $(\gamma_n^{-1} o)$ tends to ω. Assuming that $\gamma_n^{-1} o \not\to \omega$, taking a subsequence if necessary we may assume that $\gamma_n^{-1} o \to \omega'$ for some $\omega' \neq \omega$. Let $N = d(o,z)$, where z is the confluent of $[o,\omega)$ and $[o,\omega')$. Let $y \in [o,\omega)$ and $y' \in [o,\omega')$, with $d(o,y) = d(o,y') = m > N$. For some n_m we have $x_n, \gamma_n x_n \in T_o(y)$ and $\gamma_n^{-1} o \in T_o(y')$ for all $n \geq n_m$. As the γ_n's are distinct, we know that $d(\gamma_n^{-1} o, o) \to \infty$, and so we may assume that also $d(\gamma_n y', o) \geq m$ for all $n \geq n_m$. Assume that $n \geq n_m$. Then y' lies on the geodesic $[\gamma_n^{-1} o, x_n]$, and so $\gamma_n y'$ lies on the geodesic $[o, \gamma_n x_n]$. But y lies on $[o, \gamma_n x_n]$, and $d(\gamma_n y', o) \geq m = d(y,o)$. Thus $y \in [o, \gamma_n y']$. Also, y' lies on the geodesic $[o, \gamma_n^{-1} o]$, and so $\gamma_n y'$ lies on the geodesic $[o, \gamma_n o]$. Thus $y \in [o, \gamma_n o]$, i.e., $\gamma_n o \in T_o(y)$, for all $n \geq n_m$. This shows that $\gamma_n o \to \omega$. □

Let λ_Γ denote the left regular representation of Γ. If $n = 1, 2, \dots, \infty$, let $n\lambda_\Gamma$ denote the sum of n copies of λ_Γ. If σ_1 and σ_2 are two unitary representations of Γ, we write $\sigma_1 \leq \sigma_2$ if σ_1 is equivalent to a subrepresentation of σ_2.

Lemma 5.2. *Let σ denote a special or cuspidal irreducible unitary representation of G. Then the restriction $\sigma_{|\Gamma}$ of σ to Γ satisfies $\sigma_{|\Gamma} \leq \infty\lambda_\Gamma$.*

Proof. By [3, Lemmas III.2.5 and III.3.12] there is a finite complete subtree $\mathfrak{x} \subset T$ such that $|\mathfrak{x}| \geq 2$, and there is a closed subspace M_σ of $L^2(G)$, consisting of right U-invariant functions and invariant under the left regular representation λ_G of G, such that σ is equivalent to the restriction of λ_G to M_σ. Here $U = \{g \in G : gx = x \text{ for all } x \in \mathfrak{x}\}$.

Let $\{g_\alpha : \alpha \in A\}$ be a set of double coset representatives for $\Gamma\backslash G/U$. For $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 g_\alpha U = \gamma_2 g_\alpha U$ if and only if $\gamma_2^{-1}\gamma_1 \in \Gamma_\alpha$, where $\Gamma_\alpha = \Gamma \cap g_\alpha U g_\alpha^{-1}$. As Γ is discrete and as U is compact, Γ_α is finite. Let dg denote a Haar measure on G. Then for $f \in M_\sigma$ we have

$$\begin{aligned}\int_G |f(g)|^2\, dg &= \sum_\alpha \int_{\Gamma g_\alpha U} |f(g)|^2\, dg \\ &= \sum_\alpha \frac{1}{|\Gamma_\alpha|} \sum_{\gamma\in\Gamma} \int_{\gamma g_\alpha U} |f(g)|^2\, dg \\ &= \sum_\alpha \frac{m(U)}{|\Gamma_\alpha|} \sum_{\gamma\in\Gamma} |f(\gamma g_\alpha)|^2,\end{aligned}$$

where $m(U)$ denotes the Haar measure of U. Writing $f_\alpha(\gamma) = (m(U)/|\Gamma_\alpha|)^{1/2} f(\gamma g_\alpha)$ for $\alpha \in A$ and $\gamma \in \Gamma$, we see that $f \mapsto (f_\alpha)_{\alpha\in A}$ is an isometric embedding of M_σ into the direct sum of $n = \mathrm{Card}(A)$ copies of $\ell^2(\Gamma)$ which intertwines $\sigma_{|\Gamma}$ and $n\lambda_\Gamma$. □

In the next lemma, assume that s is a parameter corresponding to either the principal or the complementary series (with $s \notin \{\pm\sqrt{q}, \pm 1/\sqrt{q}\}$). In the former case, we write $\langle f, g\rangle_s$ for $\langle f, g\rangle$ defined in (1.4). If $d(o,x) = n \geq 1$, define $\xi_x = N_n \mathbf{1}_x - N_{n-1}\mathbf{1}_{x'}$, where x' is the vertex on $[o,x]$ at distance 1 from x, and where $\mathbf{1}_x$ is the indicator function of $\Omega_o(x)$, as before. In the complementary series case, we shall need below the following fact about the inner product $\langle\cdot,\cdot\rangle_s$ mentioned in Section 1: for $f, g \in \mathcal{K}(\Omega)$, $\langle f, g\rangle_s = \langle f, J_s g\rangle$ where $J_s : \mathcal{K}(\Omega) \to \mathcal{K}(\Omega)$ is a linear map such that $J_s\mathbf{1} = \mathbf{1}$ and $J_s\xi_x = j_n(s)\xi_x$ if $d(o,x) = n \geq 1$. Here $j_n(s) = s^{2(n-1)}(qs^2 - 1)/(q - s^2) > 0$ (see [3, p. 45], where the definition of ξ_x and the proof of Lemma 3.2 need correcting).

Lemma 5.3. *Suppose that x and u are vertices distinct from o, and let $g \in G$.*

(a) *If*
$$\langle \pi^s(g)\mathbf{1}, \xi_x\rangle_s \neq 0,$$
then x' lies on $[o, go]$.

(b) *If*
$$\langle \pi^s(g)\xi_x, \xi_u\rangle_s \neq 0,$$
then either
 (i) *$gx' = u'$, or*
 (ii) *gx' and u' lie on the geodesic $[o, go]$.*

Proof. (a) In the principal series case, $\langle \pi^s(g)\mathbf{1}, \xi_x\rangle_s$ equals
$$\langle \pi^s(g)\mathbf{1}, \xi_x\rangle = \int_\Omega \Big(\frac{s}{\sqrt{q}}\Big)^{-\delta(o,go,\omega)} \xi_x(\omega)\, d\nu_o(\omega),$$
while in the complementary series case, $\langle \pi^s(g)\mathbf{1}, \xi_x\rangle_s$ is a multiple of this integral, because of the form of $\langle\cdot,\cdot\rangle_s$ noted above. If $x' \notin [o, go]$, then $\delta(o, go, \omega)$ is constant on the support $\Omega_o(x')$ of ξ_x. Hence the integrand is a constant multiple of $\xi_x(\omega)$, and so the integral is 0. This proves (a).

(b) As in (a), $\langle \pi^s(g)\xi_x, \xi_u\rangle_s$ is a multiple of
$$\langle \pi^s(g)\xi_x, \xi_u\rangle = \int_\Omega \Big(\frac{s}{\sqrt{q}}\Big)^{-\delta(o,go,\omega)} \xi_x(g^{-1}\omega)\xi_u(\omega)\, d\nu_o(\omega). \tag{5.1}$$
Suppose that $gx' \notin [o, go]$. Let $n = d(o, go)$ and $m = d(o,x)$. Let $v_0 = o, \dots, v_n = go$ be the geodesic $[o, go]$ from o to go, and suppose that v_j is the last vertex common to $[o, go]$ and $[o, gx']$. Then $g\Omega_o(x) = \Omega_o(gx)$, $g\Omega_o(x') = \Omega_o(gx')$, $(gx)' = gx'$ and $d(o, gx) = m - n + 2j$. It follows that $\xi_x(g^{-1}\omega) = q^{n-2j}\xi_{gx}(\omega)$. Also, if $\xi_x(g^{-1}\omega) \neq 0$, then $g^{-1}\omega \in \Omega_o(x')$, so that $\omega \in \Omega_o(gx')$ and $-\delta(o, go, \omega) = n - 2j$.

Hence the integral in (5.1) equals
$$\Big(\frac{s}{\sqrt{q}}\Big)^{n-2j} q^{n-2j} \int_\Omega \xi_{gx}(\omega)\xi_u(\omega)\, d\nu_o(\omega).$$
For vertices $u, v \neq o$, $\int_\Omega \xi_v(\omega)\xi_u(\omega)\, d\nu_o(\omega) = 0$ unless $v' = u'$. So if also $gx' \neq u'$, then the integral in (5.1) is zero. Also, if $u' \notin [o, go]$ and $gx' \neq u'$, then $g^{-1}u' \notin [o, g^{-1}o]$ and $g^{-1}u' \neq x'$, and as $\langle \pi^s(g)\xi_x, \xi_u\rangle_s = \langle \xi_x, \pi^s(g^{-1})\xi_u\rangle_s$, this expression is zero for the same reasons. This completes the proof of (b). □

Lemma 5.4. *There exist $a, b \in T \setminus \{o\}$ such that $[o,a] \cap [o,b] = \{o\}$ and $\Gamma_a \cap \Gamma_b = \{1\}$.*

Proof. Suppose that there are no such a, b.

1. We first show that there exist $y \in T$ and $g \in \Gamma_o \setminus \{1\}$ such that $gv = v$ for all $v \in T_o(y)$. To see this, let x_0, x_1 be distinct neighbours of o. By our hypothesis, there exists $g_1 \in \Gamma_{x_0} \cap \Gamma_{x_1} \setminus \{1\}$. Notice that g_1 must fix each vertex on the geodesic $[x_0, x_1]$, and so $g_1 \in \Gamma_o$. If $g_1 v = v$ for all $v \in T_o(x_1)$, then take $y = x_1$ and $g = g_1$. Otherwise, choose $x_2 \in T_o(x_1)$ such that $g_1 x_2 \neq x_2$. Next choose $g_2 \in \Gamma_{x_0} \cap \Gamma_{x_2} \setminus \{1\}$. Notice that $g_2 \neq g_1$ and that $g_2 \in \Gamma_o$. We continue in this way, choosing $x_1, x_2, \ldots \in T$ and $g_1, g_2, \ldots \in \Gamma_o \setminus \{1\}$ such that $x_{i+1} \in T_o(x_i)$, $g_i x_i = x_i$ and $g_i x_{i+1} \neq x_{i+1}$ for each i. As g_i must fix all vertices on the geodesic $[o, x_i]$, it fixes $x_1, \ldots, x_i$, and so $g_i \neq g_1, \ldots, g_{i-1}$. As Γ_o is finite, this procedure must stop, and we can take $y = x_i$ and $g = g_i$ if it stops at the i-th step.

2. Now let y and g be as in step 1. As $g \neq 1$, there is a vertex $c \in T$ such that $gc \neq c$. Because of Lemma 5.1, there is a sequence (γ_n) of distinct elements of Γ such that $\gamma_n o \in T_o(y)$ for each n. Then $\gamma_n^{-1} g \gamma_n \in \Gamma_o$, as g fixes $\gamma_n o$. Now Γ_o is finite, and so there must exist n_0 such that $\gamma_n^{-1} g \gamma_n = \gamma_{n_0}^{-1} g \gamma_{n_0}$ for infinitely many n. Choose such an n satisfying also $d(c, \gamma_{n_0} o) \leq d(y, \gamma_n o)$. Now g commutes with $\gamma = \gamma_n \gamma_{n_0}^{-1}$. So $g\gamma c = \gamma g c \neq \gamma c$. But

$$d(\gamma c, \gamma_n o) = d(c, \gamma_{n_0} o) \leq d(y, \gamma_n o),$$

which implies that $\gamma c \in T_o(y)$. This is a contradiction, as g fixes each vertex in $T_o(y)$.

We conclude that a, b must exist with the desired properties. □

In the next lemma, let s_1 and s_2 be parameters corresponding to either the principal or the complementary series (with $s_1, s_2 \notin \{\pm\sqrt{q}, \pm 1/\sqrt{q}\}$). Let $\pi = \pi^{s_1} \otimes \pi^{s_2}$, and let H_2 be the orthogonal complement in H_{s_1,s_2} of the closed linear span H_1 of $\{\pi(g)(\mathbf{1} \otimes \mathbf{1}) : g \in G\}$.

Lemma 5.5. *There are sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ of vertices such that*

$$\xi_{x_i} \otimes \xi_{y_i} \in H_2 \quad \text{for each } i \tag{5.2}$$

and

$$\langle \pi(\gamma)(\xi_{x_i} \otimes \xi_{y_i}), \xi_{x_j} \otimes \xi_{y_j} \rangle_{s_1,s_2} = C_i \delta_1(\gamma) \delta_{i,j} \tag{5.3}$$

for all $\gamma \in \Gamma$ and $i, j = 1, 2, \ldots$, where $C_i = \langle \xi_{x_i}, \xi_{x_i} \rangle_{s_1} \langle \xi_{y_i}, \xi_{y_i} \rangle_{s_2} > 0$.

Proof. We shall choose certain points x'_j and y'_j by induction. Then any points $x_j \in T_o(x'_j)$ and $y_j \in T_o(y'_j)$ with $d(x_j, x'_j) = 1 = d(y_j, y'_j)$ will have the desired properties. Notice that we need only check (5.3) for $i \leq j$, because π is unitary.

Let a, b be as in Lemma 5.4. If $F \subset T$ is finite, we can choose $\gamma \in \Gamma$ such that $\gamma x \in T_o(a)$ for each $x \in F$. For if $M = \max\{d(o, x) : x \in F\}$, we can, by Lemma 5.1, choose $\gamma \in \Gamma$ such that $\gamma o \in T_o(a)$ and $d(\gamma o, a) \geq M$. For $x \in F$, $d(\gamma x, \gamma o) = d(x, o) \leq M$, and so $\gamma x \in T_o(a)$.

We apply this to get $\alpha_1 \in \Gamma$ such that $\alpha_1 o \in T_o(a)$. Let $x'_1 = \alpha_1 o$. Similarly, there exists $\beta_1 \in \Gamma$ such that $\beta_1 o \in T_o(b)$. We set $y'_1 = \beta_1 o$. First notice that, by Lemma 5.3(a), if $g \in G$ then $\langle \pi(g)(\mathbf{1} \otimes \mathbf{1}), \xi_{x_1} \otimes \xi_{y_1} \rangle_{s_1,s_2} \neq 0$ implies that $x'_1, y'_1 \in [o, go]$, which is impossible, as $[o, a] \cap [o, b] = \{o\}$. Hence $\xi_{x_1} \otimes \xi_{y_1} \in H_2$. By Lemma 5.3(b), if $\gamma \in \Gamma$, then $\langle \pi(\gamma)(\xi_{x_1} \otimes \xi_{y_1}), \xi_{x_1} \otimes \xi_{y_1} \rangle_{s_1,s_2} \neq 0$ implies that either (i) $\gamma x'_1 = x'_1$ and $\gamma y'_1 = y'_1$, (ii) $\gamma x'_1 = x'_1$ and $\gamma y'_1, y'_1 \in [o, \gamma o]$, (iii) $\gamma y'_1 = y'_1$ and $\gamma x'_1, x'_1 \in [o, \gamma o]$, or (iv) $\gamma x'_1$, x'_1, $\gamma y'_1$ and y'_1 all lie on $[o, \gamma o]$.

If (i) holds, then $\gamma \in \Gamma_a \cap \Gamma_b = \{1\}$. Also, (iv) cannot hold, because $[o,a] \cap [o,b] = \{o\}$ implies that $[o,x_1'] \cap [o,y_1'] = \{o\}$. Suppose that (ii) holds. Then $\gamma x_1' = x_1'$ implies that $d(x_1',o) = d(x_1',\gamma o)$ and that the geodesic $[o,x_1']$ branches off from the geodesic $[o,\gamma o]$ at a point which is exactly half way between o and γo. Then $y_1' \in [o,\gamma o]$ and $[o,a] \cap [o,b] = \{o\}$ imply that $\gamma o = o = b$, contrary to the choice of b. So (ii) is impossible. Similarly, (iii) is impossible. So (5.3) holds for $i = j = 1$.

Now suppose that $r \geq 2$, and that $\alpha_1, \dots, \alpha_{r-1}, \beta_1, \dots, \beta_{r-1} \in \Gamma$ have been found so that vertices x_i, y_i corresponding to $x_i' = \alpha_i \cdots \alpha_1 o$ and $y_i' = \beta_i \cdots \beta_1 o$ satisfy (5.2) and (5.3) for $i, j \leq r-1$.

We now apply the above remark to the finite set

$$F = \{\alpha_{r-1} \cdots \alpha_1 k (\alpha_i \cdots \alpha_1)^{-1} o : 0 \leq i \leq r-1 \text{ and } k \in \Gamma_o\},$$

to obtain $\alpha_r \in \Gamma$ such that $\alpha_r x \in T_o(a)$ for all $x \in F$. Note that $x_{r-1}' \in F$ (take $i = 0$ and $k = 1$). Let $x_r' = \alpha_r x_{r-1}' = \alpha_r \cdots \alpha_1 o$. Similarly, we can choose $\beta_r \in \Gamma$ such that $\beta_r \beta_{r-1} \cdots \beta_1 k (\beta_i \cdots \beta_1)^{-1} o \in T_o(b)$ for $0 \leq i \leq r-1$ and for $k \in \Gamma_o$. Let $y_r' = \beta_r y_{r-1}' = \beta_r \cdots \beta_1 o$.

Clearly Lemma 5.3(a) implies that $\xi_{x_r} \otimes \xi_{y_r} \in H_2$, as $x_r' \in T_o(a)$ and $y_r' \in T_o(b)$.

Now suppose that $\langle \pi(\gamma)(\xi_{x_i} \otimes \xi_{y_i}), \xi_{x_r} \otimes \xi_{y_r} \rangle_{s_1,s_2} \neq 0$ for some $i \leq r-1$. By Lemma 5.3(b) either (i) $\gamma x_i' = x_r'$ and $\gamma y_i' = y_r'$, (ii) $\gamma x_i' = x_r'$ and $\gamma y_i', y_r' \in [o,\gamma o]$, (iii) $\gamma y_i' = y_r'$ and $\gamma x_i', x_r' \in [o,\gamma o]$, or (iv) $\gamma x_i'$, x_r', $\gamma y_i'$ and y_r' all lie on $[o,\gamma o]$. Possibility (iv) is excluded, because $x_r' \in T_o(a)$ and $y_r' \in T_o(b)$. If (ii) holds, then $\gamma x_i' = x_r'$, and so $\gamma = \alpha_r \cdots \alpha_1 k (\alpha_i \cdots \alpha_1)^{-1}$ for some $k \in \Gamma_o$. Hence $\gamma o \in T_o(a)$ by the choice of α_r. So $y_r' \in [o,\gamma o]$ is impossible, because $y_r' \in T_o(b)$. So (ii) is excluded. Similarly, (iii) cannot happen. If (i) holds, then the same argument shows both that $\gamma o \in T_o(a)$ and that $\gamma o \in T_o(b)$, which is impossible.

Now suppose that $\langle \pi(\gamma)(\xi_{x_i} \otimes \xi_{y_i}), \xi_{x_r} \otimes \xi_{y_r} \rangle_{s_1,s_2} \neq 0$ for $i = r$. Arguing as in the case $r = 1$, we see that (ii)–(iv) cannot happen. If (i) holds, then $\gamma \in \Gamma_{x_r'} \cap \Gamma_{y_r'} \subset \Gamma_a \cap \Gamma_b = \{1\}$, because a, b lie on the geodesic $[x_r', y_r']$. Thus (5.3) holds for $i, j \leq r$. □

Theorem 5.6. *Assume that s_1, s_2 are parameters corresponding to either the principal or the complementary series (with $s_1, s_2 \notin \{\pm\sqrt{q}, \pm 1/\sqrt{q}\}$). Then the restriction $(\pi_2)_{|\Gamma}$ to Γ of the H_2 component π_2 of $\pi = \pi^{s_1} \otimes \pi^{s_2}$ is equivalent to $\infty\lambda_\Gamma$.*

Proof. We know that π_2 is the sum of the special and cuspidal irreducible unitary representations σ_k of G, with certain multiplicities. By Lemma 5.2, for each k we have $(\sigma_k)_{|\Gamma} \leq \infty\lambda_\Gamma$. Hence $(\pi_2)_{|\Gamma} \leq \infty\lambda_\Gamma$. On the other hand, the construction in the last lemma shows that $\infty\lambda_\Gamma \leq (\pi_2)_{|\Gamma}$. By [5, Thm. 1.1], the result is proved. □

Remarks. By taking restrictions, Theorem 4.1 gives a direct integral decomposition of the restriction $(\pi_1)_{|\Gamma}$ of the H_1 component π_1 of π. According to [3, Thm. II.7.1], at least in the co-compact case (but see also [3, p. 83]), each $(\pi^s)_{|\Gamma}$ appearing in this decomposition is irreducible, except when $s = \pm i$, when $(\pi^s)_{|\Gamma}$ may be the sum of 2 irreducible components.

It is easy to see that Γ is i.c.c. (that is, each conjugacy class other than $\{1\}$ is infinite). See [2, Lemma 6.5] for a related result.

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